7.1. Linear ODE, Homogeneous, Constant Coefficient

A **linear (ordinary) differential equation** is an ODE of the form

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = g(x),$$

where f_i and g are functions of the dependent variable x. If $g(x) = 0$, then this is called a **homogeneous** linear DE; otherwise, it is called **non-homogeneous**. If all of the function f_i are constants, then this is called a **constant coefficient** linear DE.

Solutions to Linear ODEs

Theorem 7.4: For $1 \leq j \leq n$, suppose that f_j is a function in $\mathcal{C}^\infty(\mathbb{R})$.

1. The space of all solutions to

   $$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = 0 \qquad (*)$$

   is a vector subspace of $\mathcal{C}^\infty(\mathbb{R})$.
2. Suppose also that g is a function in $\mathcal{C}^\infty(\mathbb{R})$. Then the space of all solutions to

   $$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = g(x) \qquad (**)$$

   is either the empty set, or is equal to $y_p + y_h$, where y_p is any particular solution of (**) and y_h is the full set of solutions to the associated homogeneous equation (*).

Solutions to Linear ODEs

- **Theorem 7.5:** Consider the IVP

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = g(x)$$
$$y(x_0) = c_0, y'(x_0) = c_1, \ldots, y^{(n-1)}(x_0) = c_{n-1},$$

  where every f_j and g are in $\mathcal{C}^\infty(\mathbb{R})$. This IVP has exactly one solution defined on all of $\mathbb{R}$.
- **Theorem 7.6:** Let each f_j be a function in $\mathcal{C}^\infty(\mathbb{R})$. Then the set of all solutions to the homogeneous linear ODE

$$f_n(x)y^{(n)} + f_{n-1}(x)y^{(n-1)} + \cdots + f_1(x)y' + f_0(x)y = 0$$

  is a vector subspace of $C^\infty(\mathbb{R})$ of dimension n.